University of California, Berkeley
Physics 110B, Fall 2004 (*Strovink*)
PROBLEM SET 8
Due at 5 PM on Monday, October 18, 2004

Exercises 37-41 provide exposure to multipole moments and (scalar and vector) spherical harmonics.

37.
Griffiths Problem 3.40.

38.
The electrostatic potential created by a static point charge can take a nontrivial form when the coordinate system is chosen to have an origin which, for some other reason, must be centered at point that does not coincide with the charge's position.

This problem concerns the potential $V(\vec{r})$ created by a localized charge distribution $\rho(\vec{r}\,')$. With the observation point located outside the charge distribution ($r > r'_{\rm max}$), use the standard expansion in spherical harmonics

$$\epsilon_0 V(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} \frac{Y_{lm}(\theta,\phi)}{(2l+1)r^{l+1}}\, q_{lm} \ ,$$

where the multipole moments q_{lm} are defined by

$$q_{lm} \equiv \int d\tau' \rho(\vec{r}\,') r'^{l}\, Y^*_{lm}(\theta',\phi') \, .$$

In spherical polar coordinates, consider a point charge e located at (r',θ',ϕ') with respect to a certain origin. Determine the electrostatic potential that it creates at an observation point (r,θ,ϕ), with $r > r'_{\rm max}$.
(a.)
Write down the exact value of $\mathrm{V}(r,\theta,\phi,r',\theta',\phi')$ as an infinite sum over l and m.
(b.)
Explicitly evaluating the spherical harmonics as functions of θ and ϕ (or θ' and ϕ'), write down all the monopole, dipole, and quadrupole terms ($l = 0$, 1, and 2).

39.
Arrange five finite point charges at five different positions so that all $l \le 4$ moments of the charge distribution vanish, except for the $zzzz$ component of the hexadecapole moment

$$q_{40} \equiv \int d\tau' \rho(\vec{r}\,') r'^4 Y^*_{40}(\theta',\phi') \, .$$

40.
Consider the dimensionless operator

$$\vec{L} \equiv \tfrac{1}{i}\vec{r}\times\nabla$$

(apart from a missing factor of $\hbar$, this is the same as the angular momentum operator used in quantum mechanics).
(a.)
In spherical polar coordinates, show that

$$i\vec{L} = \hat{\phi}\frac{\partial}{\partial\theta} - \frac{\hat{\theta}}{\sin\theta}\frac{\partial}{\partial\phi} \, .$$

(b.)
Express $\hat{\theta}$ and $\hat{\phi}$ in terms of $\hat{x}$, $\hat{y}$, $\hat{z}$, θ, and ϕ.
(c.)
Show that

$$iL_z = \frac{\partial}{\partial\phi}$$
$$L_\pm \equiv L_x \pm iL_y = e^{\pm i\phi}\Big(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\Big) \, .$$

[$L_\pm$ are *raising* and *lowering* operators, which, within a factor, change Y_{lm} into $Y_{l,m\pm1}$.]
(d.)
Show that

$$L^2 = L_z^2 + \tfrac{1}{2}\{L_+, L_-\} \ ,$$

where $\{a,b\}$ is the anticommutator $ab + ba$.
(e.)
Finally, show that

$$-L^2 = r^2\nabla^2_{\rm ang} \ ,$$

where ∇^2_{ang} is the part of ∇^2 which involves derivatives in θ and ϕ.

41.

Starting from the orthonormality of the spherical harmonics,

$$\int d\Omega \, Y^*_{l'm'}(\theta,\phi) \, Y_{lm}(\theta,\phi) = \delta_{ll'}\delta_{mm'} \ ,$$

and using the properties of $\vec{L}$ established in the previous problem, prove that the *vector* spherical harmonics

$$\vec{X}_{lm}(\theta,\phi) \equiv \vec{L} \, Y_{lm}(\theta,\phi)$$

satisfy the normality condition

$$\int d\Omega \, \vec{X}^*_{l'm'}(\theta,\phi) \cdot \vec{X}_{lm}(\theta,\phi) = l(l+1)\delta_{ll'}\delta_{mm'} \ .$$